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An infinite family of half-arc-transitive graphs with universal reachability relation

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ABSTRACT

The action of a subgroup G of automorphisms of a graph X is said to be *half-arc-transitive* if it is vertex-transitive and edge-transitive but not arc-transitive. In this case the graph X is said to be *G-half-arc-transitive*. In particular, when $G = \mathrm{Aut}\,X$ the graph X is said to be *half-arc-transitive*. Two oppositely oriented digraphs may be associated with any such graph in a natural way. The reachability relation of a graph admitting a half-arc-transitive group action is an equivalence relation defined on either of these two digraphs as follows. An arc e is *reachable* from an arc e' if there exists an alternating path starting with e and ending with e'. The reachability relation is clearly universal for all vertex-primitive half-arc-transitive graphs. The smallest known vertex-primitive half-arc-transitive graphs have valency 14 and no such graphs of valency smaller than 10 exist. The natural framework for the question of the existence of half-arc-transitive graphs with universal reachability relation is therefore the family of vertex-imprimitive half-arc-transitive graphs, and in particular those of valency less than 14. It is known that no such graph of valency 4 exists (see D. Marušič, Half-transitive group actions on finite graphs of valency 4, J. Combin. Theory Ser. B 73 (1998) 41–76). In this paper an infinite family of vertex-imprimitive half-arc-transitive graphs of valency 12 with universal reachability relation is constructed. These graphs have a solvable automorphism group but are not metacirculants.

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1. Introductory remarks and motivation

Throughout this paper graphs are finite, simple, connected and undirected (but with an implicit orientation of the edges when appropriate), and groups are finite. Given a graph X we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\,X$ be the set of its vertices, edges, arcs and the automorphism group of X, respectively. A subgroup $G \le \mathrm{Aut}\,X$ is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if it acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. Moreover, a subgroup $G \le \mathrm{Aut}\,X$ is said to be *half-arc-transitive* if it is vertex-transitive and edge-transitive but not arc-transitive. A graph X is said to be *G-vertex-transitive*, *G-edge-transitive*, *G-arc-transitive* and *G-half-arc-transitive* if the subgroup $G \le \mathrm{Aut}\,X$ is vertex-transitive, edge-transitive, arc-transitive, and half-arc-transitive, respectively. When $G = \mathrm{Aut}\,X$ the symbol G is omitted in all of the above notation.

A connected arc-transitive graph is also vertex-transitive and edge-transitive. The converse, however, is not true in general. In [5] Bouwer gave a construction of a half-arc-transitive graph of valency $2k$ for any integer $k \ge 2$. (Note that the smallest half-arc-transitive graph is the Doyle–Holt graph [2,8,9,13], a quartic graph of order 27.)

Graphs admitting half-arc-transitive group actions are in a one-to-one correspondence with the so-called orbital graphs of groups with non-self-paired orbitals, whereas graphs admitting arc-transitive group actions are in a one-to-one correspondence with orbital graphs of groups with self-paired orbitals. In particular, let G be a transitive permutation group acting on a set V, and let $\mathcal{O} \ne \{(v, v) \mid v \in V\}$ be a nontrivial orbital of G on V, that is, an orbit in the induced action of G on $V \times V$. Then the group G acts half-arc-transitively on the corresponding orbital graph with vertex set V and edge set $\{uv \mid (u, v) \in \mathcal{O}\}$ when $\mathcal{O} \ne \mathcal{O}^*$ and it acts arc-transitively when $\mathcal{O} = \mathcal{O}^*$, where $\mathcal{O}^* = \{(u, v) \mid (v, u) \in \mathcal{O}\}$ is the paired orbital of $\mathcal{O}$.

Half-arc-transitive graphs, quartic half-arc-transitive graphs in particular, and graphs admitting half-arc-transitive group actions in general have recently been an active topic of research. In particular, a classification of certain restricted families and various constructions of new families of such graphs together with some structural properties are known; see [3,7,10,15,16,18,19,21–23,27–33,36]. There are several approaches used in that respect, ranging from ones more algebraic in nature, such as the investigation of (im)primitivity (of half-arc-transitive group actions on graphs), to those which are more geometric and combinatorial in nature, such as the reachability relation approach, explained below.

Given a graph X of valency $2k$, $k \ge 2$, admitting a half-arc-transitive action of a subgroup $G \le \mathrm{Aut}\,X$ we let $D_G(X)$ be one of the two oppositely oriented digraphs associated with X with respect to the action of G. (An orientation on an arbitrary edge is chosen, and then the action of G defines the orientations of the remaining edges. In fact these two digraphs are the two paired orbital digraphs associated with G.) An arc e of $D_G(X)$ is *reachable* from an arc e' of $D_G(X)$ if there exists an alternating path starting with e and ending with e'. This equivalence relation is called the *reachability relation* (see [6] and [20] where this concept is considered in a larger context of infinite graphs). The corresponding equivalence classes are called *alternating cycles* when X is of valency 4 (see [21]), and *alternets* in the general case (see [35]). The concept of alternets is equivalent to the concept of kernels introduced in [24]. We perform the following operation on $D_G(X)$. First we let each vertex $v \in V(X)$ split into a vertex v^+ which keeps all incoming arcs and a vertex v^- which keeps all outgoing arcs, obtaining a possibly disconnected digraph, whose underlying graph, denoted by X_G^* (and X^* when $G = \mathrm{Aut}\,X$), is of valency k and of order $2|V(X)|$. (Observe that for any component Y of X_G^*, the constituent $G^{V(Y)}$ acts transitively on $V(Y)$.) One can easily see that there are isomorphic copies of components of X_G^* inside X. Any such component (or its isomorphic counterpart in X) is called a *G-kernel of X*, and is denoted by $\mathrm{Ker}_G(X)$ (for details see [24]). Of course, the above reachability relation on $D_G(X)$ is universal if and only if the graph X_G^* is connected. In other words, the reachability relation on $D_G(X)$ is universal, and hence $D_G(X)$ itself is an alternet, if and only if the kernel $\mathrm{Ker}_G(X)$ coincides with $D_G(X)$. The concepts of alternets and kernels may also be translated into the language of the so-called *alter-sequences* associated with a similar equivalence relation defined on the vertex set of a graph admitting a half-arc-transitive group action (see [25]).

In view of the above comments it follows that when the reachability relation on $D_G(X)$ is not universal the corresponding alternets give rise to a decomposition of $E(X)$. The study of the structure of such graphs can then be performed making use of various interactions of alternets. For example the so-called attachment number, that is, the size of the intersection of two alternets in a graph, has proved to be an important parameter of a graph in question. On the other hand, when the reachability relation is universal no such geometric or combinatorial tools are at hand. It is therefore essential to first study the basic objects, that is, the graphs admitting half-arc-transitive group action with universal reachability relation, if one is after a classification of the whole class of such graphs.

For example, the situation is well understood for graphs of valency 4. Namely, there are no quartic graphs with universal reachability relation, whereas a graph with two alternating cycles is necessarily isomorphic to $\mathrm{Cay}(\mathbb{Z}_n, \{\pm 1, \pm r\})$, where $r^2 \equiv \pm 1 \pmod{n}$ (see [21]). This leaves us with the graphs having at least three alternating cycles (see [21,26,27,29,30] for some of the results and ongoing research in this context).

The reachability relation arising from a vertex-primitive half-arc-transitive group action in a graph is clearly universal. (For a graph X and a transitive subgroup $G \leq \mathrm{Aut}\,X$, a partition $\mathcal{B}$ of $V(X)$ is *G-invariant* if the elements of G permute the parts of $\mathcal{B}$ setwise. If $\{V(X)\}$ and $\{\{v\} \mid v \in V(X)\}$ are the only G-invariant partitions of $V(X)$, then X is said to be *G-vertex-primitive*, and it is said to be *G-vertex-imprimitive* otherwise. If $G = \mathrm{Aut}\,X$ we simply say that X is vertex-primitive and vertex-imprimitive, respectively.) In [15] it was shown that there is no vertex-primitive half-arc-transitive graph of valency less than 10. Moreover in [15,17] two infinite families of vertex-primitive half-arc-transitive graphs were constructed; the smallest valency of graphs in both of these families is 14. The smallest two valencies for which the existence of vertex-primitive half-arc-transitive graphs has not yet been settled are thus 10 and 12.

In view of the fact that universality of the reachability relation is given for free in the case of vertex-primitive graphs admitting half-arc-transitive group actions, the appropriate context within which the reachability relation needs to be studied is provided by vertex-imprimitive graphs admitting half-arc-transitive group actions. In this paper an infinite family of vertex-imprimitive half-arc-transitive graphs of valency 12 with universal reachability relation is constructed (see Theorem 2.2). These graphs have solvable automorphism groups. Also, these graphs are not metacirculants adding to the importance of this construction in view of the fact that most of the known vertex-imprimitive half-arc-transitive graphs are metacirculants. (Following [1], an (m, n)-*metacirculant*, $m \geq 1$ and $n \geq 2$, is a graph admitting a transitive group generated by an (m, n)-semiregular automorphism ρ – that is, an automorphism with m orbits of length n and no other orbits – normalized by an automorphism σ cyclically permuting the orbits of ρ in such a way that σ^m has at least one fixed vertex.) For example, every half-arc-transitive graph of order $3p$, p a prime, is a metacirculant [3]. Also, all half-arc-transitive graphs of order $4p$, p a prime, and of valency 4 or 6, are metacirculants [11,34]. It is therefore worth mentioning that the above family of graphs contains previously unknown half-arc-transitive graphs of order $4p$, p a prime, which are not metacirculants, with $p = 73$ providing the smallest example.

2. The construction

Let $r > 1$ be an integer and let $n > 3$ be a divisor of $r^2 + r + 1$. Then define $X_1(r; n)$ to be the graph with vertex set $V(X_1(r; n)) = \mathbb{Z}_4 \times \mathbb{Z}_n$ with adjacencies in $X_1(r; n)$ satisfying the following conditions:

$$
\begin{aligned}
(0,i) \sim (0,j) &\iff j-i \in \{\pm 1, \pm r, \pm r^2\},\\
(0,i) \sim (1,j) &\iff j-i \in \{-1, r^2\},\\
(0,i) \sim (2,j) &\iff j-i \in \{1, -r\},\\
(0,i) \sim (3,j) &\iff j-i \in \{r, -r^2\},\\
(1,i) \sim (1,j) &\iff j-i \in \{\pm r\},\\
(1,i) \sim (2,j) &\iff j-i \in \{\pm 1, r, r^2\},\\
(1,i) \sim (3,j) &\iff j-i \in \{-1, -r, \pm r^2\},\\
(2,i) \sim (2,j) &\iff j-i \in \{\pm r^2\},\\
(2,i) \sim (3,j) &\iff j-i \in \{1, \pm r, r^2\},\\
(3,i) \sim (3,j) &\iff j-i \in \{\pm 1\}.
\end{aligned}
$$

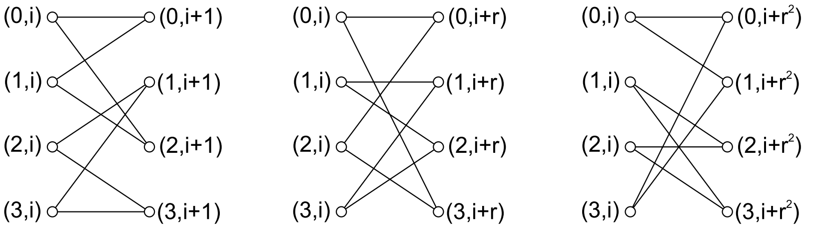

Fig. 1. The bipartite graphs $X[B_i, B_{i+1}]$, $X[B_i, B_{i+r}]$ and $X[B_i, B_{i+r^2}]$ in $X_1(r; n)$.

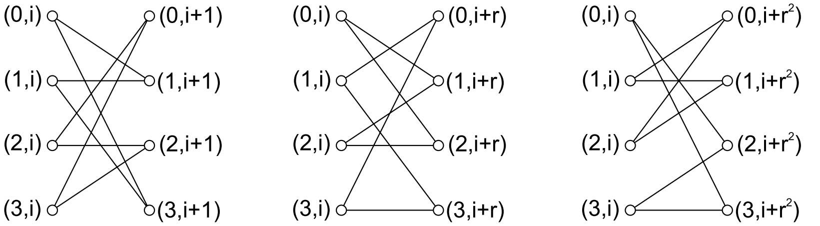

Fig. 2. The bipartite graphs $X[B_i, B_{i+1}]$, $X[B_i, B_{i+r}]$ and $X[B_i, B_{i+r^2}]$ in $X_2(r; n)$.

Similarly, let $X_2(r; n)$ be the graph with vertex set $V(X_1(r; n)) = \mathbb{Z}_4 \times \mathbb{Z}_n$ and adjacencies in $X_2(r; n)$ satisfying the following conditions:

$$\begin{array}{lll}
(0, i) \sim (1, j) & \iff & j - i \in \{1, \pm r, -r^2\}, \\
(0, i) \sim (2, j) & \iff & j - i \in \{-1, r, \pm r^2\}, \\
(0, i) \sim (3, j) & \iff & j - i \in \{\pm 1, -r, r^2\}, \\
(1, i) \sim (1, j) & \iff & j - i \in \{\pm 1, \pm r^2\}, \\
(1, i) \sim (2, j) & \iff & j - i \in \{-r, -r^2\}, \\
(1, i) \sim (3, j) & \iff & j - i \in \{1, r\}, \\
(2, i) \sim (2, j) & \iff & j - i \in \{\pm 1, \pm r\}, \\
(2, i) \sim (3, j) & \iff & j - i \in \{-1, -r^2\}, \\
(3, i) \sim (3, j) & \iff & j - i \in \{\pm r, \pm r^2\}.
\end{array}$$

Let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_n\}$ where $B_i = \{(0, i), (1, i), (2, i), (3, i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_n$ and let $i \in \mathbb{Z}_n$. Then for each $j \in \mathbb{Z}_n, j \neq i$, the subgraph $X[B_i, B_j]$ induced on the union $B_i \cup B_j$ is not an independent set of vertices if and only if $j \in \{i \pm 1, i \pm r, i \pm r^2\}$. Moreover, for each such j we have that $X[B_i, B_j] \cong 2C_4$; see Figs. 1 and 2. Observe that r is an element of order 3 in $\mathbb{Z}_n^*$, that $X_1(r; n)$ and $X_2(r; n)$ are both of valency 12, and that the edge-disjoint union $X_1(r; n) \cup X_2(r; n)$ is the wreath product $Y \wr 4K_1$ where $Y = \mathrm{Cay}(\mathbb{Z}_n, \{\pm 1, \pm r, \pm r^2\})$. (The *wreath product* $X \wr Y$, sometimes also called the *lexicographic product*, of a graph X with a graph Y has vertex set $V(X) \times V(Y)$, with two vertices (a, u) and (b, v) adjacent if $ab \in E(X)$ or if $a = b$ and $uv \in E(Y)$.)

The following proposition is a direct consequence of the fact that r is of order 3 in $\mathbb{Z}_n^*$. The proof is left to the reader.

Proposition 2.1. *Let r be an integer, let $n > 3$ be a divisor of $1 + r + r^2$, and let $X \in \{X_1(r; n), X_2(r; n)\}$. Then the permutations ρ and ψ of $V(X)$ defined, respectively, by $(i, j)^\rho = (i, j + 1)$ and $(i, j)^\psi = (i^\delta, jr)$, where $\delta = (0)(1\ 2\ 3) \in S_4$ and $i, j \in \mathbb{Z}_n$, are automorphisms of the graph X. In addition, ρ is of order n and ψ is of order 3.*

Note that the above automorphism ρ is $(4, n)$-semiregular, and therefore the graphs $X_1(r; n)$ and $X_2(r; n)$ can be presented in the so-called Frucht notation [12] as illustrated in Fig. 3.

Let $H = \{1, \tau_1, \tau_2, \tau_3\}$ be the Klein 4-group acting on $\mathbb{Z}_4$ so that $\tau_1 = (0\,1)(2\,3)$, $\tau_2 = (0\,2)(1\,3)$ and $\tau_3 = (0\,3)(1\,2)$. Let $X \in \{X_1(r; n), X_2(r; n)\}$ and let K be the kernel of the action of $\mathrm{Aut}\,X$

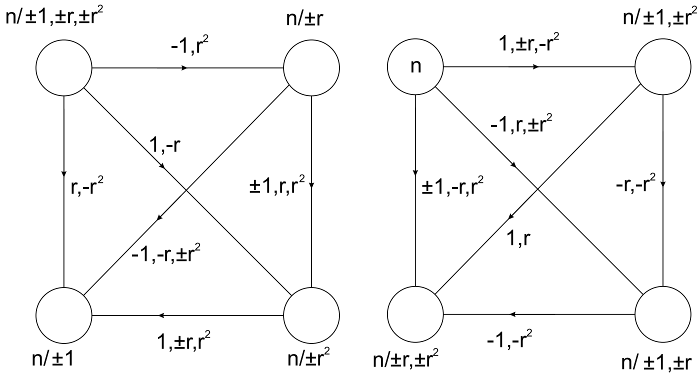

Fig. 3. The graph $X_1(r; n)$ in the left-hand side picture and the graph $X_2(r; 4n)$ in the right-hand side picture are given in Frucht's notation, relative to a $(4, n)$-semiregular automorphism.

on $\mathcal{B}$. We shall be sloppy and shall identify restrictions of elements of K to sets B_i by elements of H. For instance, when we say that the restriction γ_i of $\gamma \in K$ to B_i is, for example, τ_1, we mean that $\gamma_i = ((0, i)(1, i))((2, i)(3, i))$. Now, the structure of X indicated by Figs. 1 and 2 implies that the restrictions γ_i must satisfy the following conditions:

$$\gamma_i \in \{1, \tau_1\} \iff \gamma_{i+1} \in \{1, \tau_2\} \quad \forall i \in \mathbb{Z}_n, \tag{1}$$

$$\gamma_i \in \{1, \tau_2\} \iff \gamma_{i+r} \in \{1, \tau_3\} \quad \forall i \in \mathbb{Z}_n, \tag{2}$$

$$\gamma_i \in \{1, \tau_3\} \iff \gamma_{i+r^2} \in \{1, \tau_1\} \quad \forall i \in \mathbb{Z}_n. \tag{3}$$

We are now ready to prove the main theorem of the paper.

Theorem 2.2. *Let $k \geq 1$, let $r = 2^k$ and let $i \in \{1, 2\}$. Then the graph $X = X_i(r; r^2 + r + 1)$ is vertex-imprimitive and is half-arc-transitive unless $k = 1$, in which case X is arc-transitive. Moreover, the graph X has universal reachability relation if and only if $r \equiv 2 \pmod 3$ (that is, if k is odd).*

Proof. Let $n = r^2 + r + 1$ and let $G \leq \mathrm{Aut}\,X$ be the largest subgroup of $\mathrm{Aut}\,X$ such that the set $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_n\}$ is a G-invariant partition of $V(X)$. Observe that the automorphisms ρ and ψ, defined in Proposition 2.1, both belong to G. The proof consists of three steps. In the first step we give an explicit construction of a nontrivial involution $\sigma \in K$ where K is the kernel of the action of G on the set $\mathcal{B}$, the existence of which implies vertex-transitivity and edge-transitivity of X. In the second step we prove that $G = \mathrm{Aut}\,X$ and that X is half-arc-transitive. In the last step we prove that the reachability relation on X is universal if and only if $r \equiv 2 \pmod 3$.

Step 1:

We construct $\sigma \in K$ by specifying the restrictions $\sigma_i = \sigma^{B_i}$, $i \in \mathbb{Z}_n$. We do this recursively in the following manner. For $k = 1$ (and thus $r = 2$) we let

$$\Sigma_2 = \begin{bmatrix} \tau_2 & 1 \\ \tau_1 & \tau_2 \end{bmatrix}.$$

For $r \geq 2$ we then construct Σ_{2r} recursively as

$$\Sigma_{2r} = \left[\begin{array}{c|c} \Sigma_r & J_r \\ \hline \Sigma_r^* & \Sigma_r \end{array}\right],$$

where J_r is the unit matrix (consisting of all ones) of size r and Σ_r^* is the matrix obtained from Σ_r by multiplying its first row elementwise by τ_3. For example,

$$\Sigma_2^* = \begin{bmatrix} \tau_1 & \tau_3 \\ \tau_1 & \tau_2 \end{bmatrix}.$$

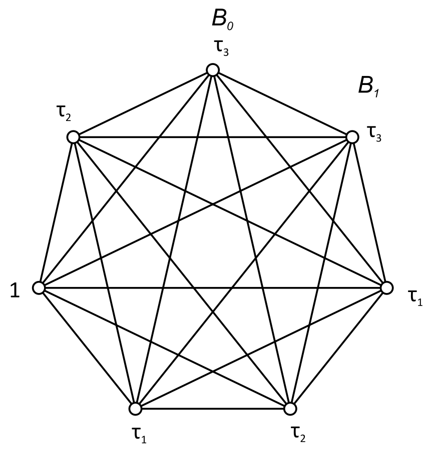

Fig. 4. The restrictions of σ to the blocks of $\mathcal{B}$ shown in the quotient graph $X_{\mathcal{B}}$ for $n = 7$.

Now, let $t_{i,j}$ be the (i, j)-th entry of Σ_r, let $\mathrm{Row}_i(\Sigma_r) = [t_{i,1}, t_{i,2}, \dots, t_{i,r}]$ and let $\mathrm{Col}_i(\Sigma_r) = [t_{1,i}, t_{2,i}, \dots, t_{r,i}]^T$. We define the automorphism σ as follows. We form a sequence of entries of Σ_r by arranging them in the lexicographic order

$$[t_{1,1}, t_{1,2}, t_{1,3}, \dots, t_{1,r}, t_{2,1}, t_{2,2}, \dots, t_{r,r}] \tag{4}$$

and then define the restrictions σ_i in the following way. We let $\sigma_0 = \tau_3$ and let σ_{-i} be the i-th entry of the sequence (4) for each $i \in \{1, 2, \dots, r^2\}$, that is,

$$\sigma_{-1} = \sigma_{r^2+r} = t_{1,1}, \qquad \sigma_{-2} = t_{1,2}, \dots, \qquad \sigma_{-r^2+1} = \sigma_{r+2} = t_{r,r-1}, \qquad \sigma_{-r^2} = \sigma_{r+1} = t_{r,r}.$$

For the remaining r restrictions $\sigma_r, \sigma_{r-1}, \dots, \sigma_1$ we let

$$\sigma_r = \tau_3 t_{1,1}, \qquad \sigma_{r-1} = \tau_3 t_{1,2}, \dots, \qquad \sigma_1 = \tau_3 t_{1,r}$$

(see also Figs. 4 and 5).

Note that $\sigma \in \mathrm{Aut}\, X$ if and only if σ satisfies (1)–(3). Moreover, one can see that the conditions (1)–(3) all hold if and only if (1) and

$$\sigma_{i-r}\sigma_i\sigma_{i+1} = 1 \quad \forall i \in \mathbb{Z}_n \tag{5}$$

hold.

We prove (1) and (5) by induction on k. For $k = 1$, and thus $r = 2$, conditions (1) and (5) can easily be verified. Now suppose that (1) and (5) hold for every $r = 2^k$, $k \geq 1$, and let us show that then they also hold for $2r = 2^{k+1}$. First observe that the condition (1) for $n = (2r)^2 + 2r + 1$ is equivalent to the following set of equations for the entries $t_{i,j}$ of Σ_{2r}:

$$\begin{aligned}
&t_{i,j} \in \{1, \tau_1\} \iff t_{i,j-1} \in \{1, \tau_2\} \quad \forall i \in \{1, 2, \dots, 2r\} \text{ and } \forall j \in \{2, 3, \dots, 2r\};\\
&t_{i,1} \in \{1, \tau_1\} \iff t_{i-1,2r} \in \{1, \tau_2\} \quad \forall i \in \{2, 3, \dots, 2r\};\\
&t_{1,1} \in \{1, \tau_1\} \iff \sigma_0 \in \{1, \tau_2\};\\
&\sigma_0 \in \{1, \tau_1\} \iff \tau_3 t_{1,2r} \in \{1, \tau_2\};\\
&\tau_3 t_{1,j} \in \{1, \tau_1\} \iff \tau_3 t_{1,j-1} \in \{1, \tau_2\} \quad \forall j \in \{2, 3, \dots, 2r\};\\
&\tau_3 t_{1,1} \in \{1, \tau_1\} \iff t_{2r,2r} \in \{1, \tau_2\}.
\end{aligned}$$

Moreover, the condition (5) for $n = (2r)^2 + 2r + 1$ is equivalent to the following set of equations for $t_{i,j}$ entries of Σ_{2r}:

$$t_{i,j}t_{i-1,j}t_{i-1,j-1} = 1 \quad \forall i \in \{2, 3, \dots, 2r\} \text{ and } \forall j \in \{2, 3, \dots, 2r\};$$

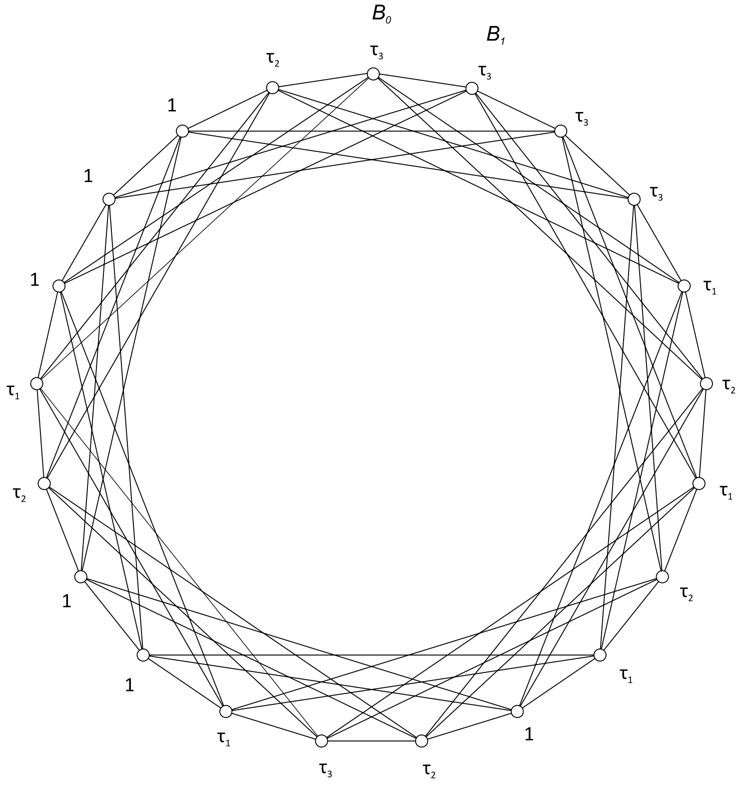

Fig. 5. The restrictions of σ to the blocks of $\mathcal{B}$ shown in the quotient graph $X_{\mathcal{B}}$ for $n = 21$.

$$\begin{aligned}
&\tau_3 t_{1,j} t_{2r,j} t_{2r,j-1} = 1 \quad \forall j \in \{2, 3, \dots, 2r\};\\
&\tau_3 t_{1,1} t_{2r,1} t_{2r-1,2r} = 1;\\
&t_{i,1} t_{i-1,1} t_{i-2,2r} = 1 \quad \forall i \in \{3, 4 \dots, 2r\};\\
&t_{1,j} \tau_3 t_{1,j+1} \tau_3 t_{1,j} = 1 \quad \forall j \in \{1, 2, \dots, 2r-1\};\\
&\sigma_0 \tau_3 t_{1,1} t_{2r,2r} = 1;\\
&t_{2,1} t_{1,1} \sigma_0 = 1;\\
&t_{1,r} \sigma_0 \tau_3 t_{1,r} = 1.
\end{aligned}$$

The induction assumptions imply that certain conditions hold for entries of Σ_{2r}. For example,

$$t_{i,j} t_{i-1,j} t_{i-1,j-1} = 1 \quad \forall i \in \{2, 3 \dots, r\} \text{ and } \forall j \in \{1, 2, \dots, r\}.$$

However, since

$$\begin{aligned}
&\mathrm{Row}_1(\Sigma_{2r}) = [\tau_2, 1, 1, \dots, 1],\\
&\mathrm{Row}_r(\Sigma_{2r}) = [\underbrace{\tau_1, \tau_2, \tau_1, \dots, \tau_2, \tau_1, \tau_2}_{r}, 1, \dots, 1],\\
&\mathrm{Row}_{r+1}(\Sigma_{2r}) = [\tau_1, \underbrace{\tau_3, \dots, \tau_3}_{r-1}, \tau_2, \underbrace{1, \dots, 1}_{r-1}],\\
&\mathrm{Row}_{2r}(\Sigma_{2r}) = [\tau_1, \tau_2, \tau_1, \dots, \tau_2, \tau_1, \tau_2],\\
&\mathrm{Col}_1(\Sigma_{2r}) = [\tau_2, \tau_1, \dots, \tau_1]^T,
\end{aligned}$$

$$\mathrm{Col}_r(\Sigma_{2r}) = [\underbrace{1, \ldots, 1}_{r-1}, \tau_2, \tau_3, \underbrace{1, \ldots, 1}_{r-2}, \tau_2]^T,$$

$$\mathrm{Col}_{r+1}(\Sigma_{2r}) = [\underbrace{1, \ldots, 1}_{r}, \tau_2, \underbrace{\tau_1, \tau_1, \ldots, \tau_1}_{r-1}]^T,$$

$$\mathrm{Col}_{2r}(\Sigma_{2r}) = [1, \ldots, 1, \tau_2]^T,$$

one can easily verify that (1) and (5) both hold, which thus implies that $\sigma \in \mathrm{Aut}\,X$. The nature of the action of σ implies that X is vertex-transitive and edge-transitive, as claimed.

For $k = 1$ (and thus $n = 7$) the fact that X is vertex-imprimitive and arc-transitive can be verified by a computer (for instance, using MAGMA software [4]). For the rest of the proof we thus assume $k \geq 2$ (and thus $n \geq 21$).

Step 2:

We simplify the notation and let $x_i = (0, i)$, $y_i = (1, i)$, $z_i = (2, i)$, $w_i = (3, i)$, for each $i \in \mathbb{Z}_n$. We first show that the set $\mathcal{B}$ is an $\mathrm{Aut}\,X$-invariant partition. By definition of the graph X (see also Figs. 1 and 2) each of the vertices y_0, z_0 and w_0 is at distance 2 from x_0 and has precisely four neighbors in common with x_0. We now show that no other vertex of X shares this property, proving that B_0 is indeed a block of imprimitivity for $\mathrm{Aut}\,X$. Suppose to the contrary that such a vertex, say v, exists. Then $v \in B_i$ for some nonzero

$$i \in \{\pm 2, \pm 2r, \pm 2r^2, \pm 1 \pm r, \pm 1 \pm r^2, \pm r \pm r^2\}. \tag{6}$$

Combining together the facts that $k > 1$, that $r^3 = 1$ and that n is odd, one can easily show that the 18 elements from (6) are pairwise distinct. As x_0 does not have more than two neighbors in a single set from $\mathcal{B}$, we cannot have that $i \in \{\pm 2, \pm 2r, \pm 2r^2\}$. Moreover, two of the four common neighbors of x_0 and v are in one set from $\mathcal{B}$, say B_j, and two in another set from $\mathcal{B}$. In addition, the fact that X is vertex-transitive and that ψ from Proposition 2.1 is an automorphism of X implies that we can assume $j = 1$. Now, if $X = X_1(r; n)$ then the two common neighbors from B_1 are x_1 and z_1. However, Fig. 1 implies that then $i = 1 + r$ and $v = x_{1+r}$. But x_{1+r} is not at distance 2 from x_0 (since $1 + r + r^2 = 0$ in $\mathbb{Z}_n$, we have $x_{1+r} \sim x_0$), a contradiction. A similar argument applies when $X = X_2(r; n)$. Hence, $\mathcal{B}$ is an $\mathrm{Aut}\,X$-invariant partition, as claimed. Note that this implies $G = \mathrm{Aut}\,X$.

Now, suppose that X is arc-transitive. Since $\mathcal{B}$ is $\mathrm{Aut}\,X$-invariant, the natural action of the group $\mathrm{Aut}\,X$ on the quotient graph $Y = X_{\mathcal{B}} \cong \mathrm{Cay}(\mathbb{Z}_n, \{\pm 1, \pm r, \pm r^2\})$ is arc-transitive as well. Using [14, Theorem 1] one can easily see that Y is a normal circulant. By [37, Proposition 1.5.] every element $\bar{\alpha} \in \mathrm{Aut}\,Y$ fixing the vertex 0 is an automorphism of the group $\mathbb{Z}_n$. This implies that an element $\bar{\alpha} \in \mathrm{Aut}\,Y$ fixing 0 and mapping 1 to -1 maps each $i \in \mathbb{Z}_n$ to $-i$. Thus, letting $\alpha \in \mathrm{Aut}\,X$ be an automorphism fixing x_0 and mapping x_1 to x_{-1} we see that α interchanges each B_i with B_{-i}. Inspecting adjacencies between vertices of the blocks B_{-1}, B_0 and B_1 we find that the only possibility for α is to fix w_0 and interchange y_0 with z_0. But inspecting adjacencies between blocks B_{-r}, B_0 and B_r we find that α would have to fix y_0. Thus, α does not exist, and so X is half-arc-transitive, as claimed.

Step 3:

Suppose first that $r \equiv 2 \pmod 3$ and let us show that in this case X has universal reachability relation. Recall that the graph X has universal reachability relation if and only if the graph X^* (defined in Section 1) is connected. Let $D(X)$ be the digraph associated with the half-arc-transitive action of $\mathrm{Aut}\,X$ on X in which x_1 is a successor of x_0. The existence of the automorphism ψ from Proposition 2.1 implies that $D(X)_{\mathcal{B}} \cong \mathrm{Cay}(\mathbb{Z}_n, \{1, r, r^2\})$. Let $B_i^+ = \{v^+ \mid v \in B_i\}$ and $B_i^- = \{v^- \mid v \in B_i\}$, where $B_i \in \mathcal{B}$ and the notation v^+, v^- is as in Section 1. One can easily see that for every $i \in \mathbb{Z}_n$ all the vertices in B_i^+ belong to the same connected component of X^* and that all the vertices in B_i^- belong to the same connected component of X^*. Let $C_i(X^*)$ be the connected component of X^* containing B_i^-.

Suppose first that $r \equiv 2 \pmod 3$. Since B_{i+r}^+, B_{i+r-1}^-, $B_{i+r-1+r^2}^+ = B_{i-2}^+$ and $B_{i+r-1+r^2-1}^- = B_{i-3}^-$ all belong to $C_i(X^*)$, it follows that B_{-3}^- also belongs to $C_0(X^*)$. Therefore, $C_0(X^*)^{\rho^{-3}} = C_0(X^*)$. (Note that each automorphism of X can be viewed as an automorphism of X^*.) Since $r \equiv 2 \pmod 3$, we have that $n = r^2 + r + 1 \equiv 1 \pmod 3$, and thus also $C_0(X^*)^{\rho} = C_0(X^*)$ holds. It follows that $C_0(X^*) = X^*$ proving that the reachability relation on X is universal.

Suppose now that $r \equiv 1 \pmod 3$. Then $n \equiv 0 \pmod 3$. Hence each of $\pm(1-r)$, $\pm(1-r^2)$, $\pm(r-r^2)$ is divisible by 3. This implies that on each alternating path starting at the edge x_0x_1 every other vertex belongs to a block B_i with $i \equiv 0 \pmod 3$ and every other vertex belongs to a block B_i with $i \equiv 1 \pmod 3$. Therefore, the edge x_1x_2 is never reached with such a path, and so the reachability relation on X is not universal (in fact, it has exactly three equivalence classes). $\square$

By the following proposition the constructed graphs $X_1(r; r^2+r+1)$ and $X_2(r; r^2+r+1)$ are not metacirculants when r is a power of 2.

Proposition 2.3. *The graphs $X_i(r; r^2 + r + 1)$, where $r = 2^k$, $k > 1$ and $i \in \{1, 2\}$, have a solvable automorphism group but are not metacirculants.*

Proof. Let $X = X_1(r; r^2 + r + 1)$ (the proof for $X = X_2(r; r^2 + r + 1)$ is analogous and is left to the reader). Let $A = \mathrm{Aut}\, X$ and let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_n\}$ where $B_i = \{(0, i), (1, i), (2, i), (3, i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_n$. From the proof of Theorem 2.2 it follows that K and A/K (where K is as in the proof of Theorem 2.2) are both solvable, and thus A is solvable.

To show that X is not a metacirculant it suffices to show that A has no transitive metacyclic subgroup. Suppose on the contrary that there is such a subgroup $G = \langle \omega, \gamma \rangle \le A$, where $H = \langle \omega \rangle$ is normal in G and $G/H \cong \langle \gamma \rangle$. Observe that A, and hence G, has no element of order 4. Let $\mathcal{C}$ be the set of orbits of H. If these orbits are either of length d or $4d$, where d is a divisor of $n = r^2 + r + 1$, then either H or the subgroup $\langle \gamma \rangle$ would contain an element of order 4, which is not possible. We may therefore assume that the orbits in $\mathcal{C}$ are of length $2d$, d a divisor of n. Clearly there exist $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $|B \cap C| = 2$. Consequently, $B \cap C$ gives rise to a G-invariant partition $\mathcal{D}$ with blocks of length 2. Observe that these blocks coincide with orbits of the semiregular element ω^d of order 2 (which is central in G) and that each block from $\mathcal{B}$ consists of two blocks from $\mathcal{D}$. Thus for any two adjacent blocks $D, D' \in \mathcal{D}$ the bipartite subgraph of X induced by $D \cup D'$ is isomorphic either to $2K_2$ or to $K_{2,2}$. Without loss of generality assume that $D_0 = \{(0, 0), (1, 0)\} \in \mathcal{D}$. (If either of the other two subsets of B_0 containing $(0, 0)$ was a block in $\mathcal{D}$, then in the argument below we would replace the automorphism mapping each B_i to B_{i+1} by its r-th or r^2-th power). Since the bipartite subgraph of X induced by two adjacent blocks is regular it follows that $D_1 = \{(0, 1), (3, 1)\}$ and $D'_1 = \{(1, 1), (2, 1)\}$ belong to $\mathcal{D}$. Using the arguments from the proof of Theorem 2.2 (recall that G is transitive on X) one can easily show that there exists an element mapping each B_i to B_{i+1}, $i \in \mathbb{Z}_n$. It follows that there is some $D_2 \in \mathcal{D}$ with $D_2 \subset B_2$ such that the bipartite subgraph of X induced by $D_1 \cup D_2$ is isomorphic to $K_{2,2}$. This is clearly impossible, which shows that a transitive metacyclic subgroup of A does not exist, as claimed. $\square$

3. Conclusions

In this paper we have constructed two vertex-imprimitive half-arc-transitive graphs of valency 12 and of order $4(r^2 + r + 1)$, for every integer $r = 2^k$, $k > 1$. Moreover, we have shown that the constructed graphs have universal reachability relation when k is an odd integer. We believe that the results obtained will prove useful in some open problems regarding half-arc-transitive graphs, such as in obtaining the classification of half-arc-transitive graphs of order $4p$, where p is a prime.

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